1. Logarithmic functions

- Logarithms
- Laws of logarithms
- Using logarithms to solve equations
- Natural logarithms
- Logarithmic functions
- Graphs of logarithmic functions
- Logarithmic scale
- Graphs with logarithmic scales
- Use of logarithmic scales
- Miscellaneous exercise one

Situation One

Various estimates could be made for how many years ago the population of the world was just one million. Now it exceeds seven billion, i.e. it now exceeds 7 000 000 000.

Discuss any difficulties a person would face if they were to try to display these world population figures, from one million to seven billion, graphically.

Situation Two

In 1982 an earthquake measuring 6.0 on the Richter scale (a scale for measuring the intensity of an earthquake) occurred in the Yemen, and is thought to have resulted in approximately 2800 deaths.

In 2010 an earthquake measuring 7.0 on the Richter scale occurred in Haiti and, according to some estimates, may have resulted in more than 300 000 deaths.

Now 7.0 is approximately 1.17×6.0 and yet the number of deaths in the Haiti earthquake far exceeds 1.17×2800!

An earthquake in Japan in 2011 measured 9.0 on the Richter scale. Despite this being higher than the Richter scale measurement for the Haiti earthquake the death toll was thought to be about 16 000. Whilst this is a tragically high number of fatalities it is well below the Haiti earthquake death toll, despite the higher rating on the Richter scale.

Discuss the above comments comparing earthquake death tolls and Richter scale readings. Do some research about the Richter scale.

Situation Three

Let us suppose that the number of cells in a colony of bacteria doubles every hour. When timing commences the colony consists of 50 cells. Assuming the doubling continues, how long will it take for the number of cells to reach ten million?

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Situation Four

An object with a temperature of 90°C is placed in an environment with temperature 20°C. The temperature of the object, t minutes later, is T°C, where T approximately follows the mathematical rule

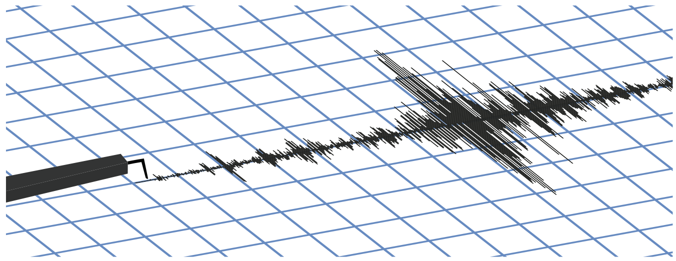
$$T = 20 + 70e^{-0.4t}.$$

After how many minutes will the temperature of the object be 35°C?

Logarithms

WS Logarithms

If you did some research about the Richter scale, as Situation Two on the previous page suggested, you probably discovered that its scale is *logarithmic* rather than linear. However this immediately raises the question: *What does it mean for a scale to be logarithmic?* We will investigate what a logarithm is in this chapter.



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Situation Three, and indeed Situation Four, involved solving an equation in which the unknown value featured as an *index* in the equation.

Situation Three required you to solve $\quad 50 \times 2^t = 10\,000\,000.$

Situation Four required you to solve $\quad 35 = 20 + 70e^{-0.4t}.$

How did you go about solving these equations?

Let us consider equations of this type further, i.e. equations in which the unknown features as an index.

Suppose we have to solve: $\quad 5 \times 2^x = 80$
Divide each side of the equation by 5. $\quad 2^x = 16$
From our knowledge of the powers of 2: $\quad x = 4$

Suppose instead we were asked to solve: $\quad 5 \times 2^x = 115$
Dividing each side of the equation by 5. $\quad 2^x = 23$

Now our knowledge of the powers of 2 tells us that x must lie between 4 ($2^4 = 16$) and 5 ($2^5 = 32$).

If we want to be more precise than this we could:

- look at the graph of $y = 2^x$ and see where it cuts the line $y = 23$.
- try some values between $x = 4$ and $x = 5$, evaluate 2^x, and adjust our trials accordingly (trial and adjust).
- use the solve facility of some calculators. Note that whilst the display below left gives the answer as 4.523 561 956 the display below right is typical of the response we would get from a calculator that has been asked to give the solution as an exact value. This exact solution is given using *logarithms*, 'ln' being the abbreviation for the 'natural logarithm' of a number.

solve($5 \times 2^x = 115, x$)
$x = 4.523561956$

solve($5 \times 2^x = 115, x$)
$x = \dfrac{\ln(23)}{\ln(2)}$

Let us now consider this idea of the *logarithm* of a number.

ISBN 9780170395205

Note: The idea of a logarithm of a number proves useful in a number of applications other than simply being able to give the exact solution to some exponential equations, as we shall see as this course continues. Hence their introduction here.

- The number 10, raised to the power 2, is equal to 100, i.e. $10^2 = 100$.
  We say that the logarithm to the **base** 10, of 100 is 2. i.e. $\log_{10} 100 = 2$.

  The logarithm to the base 10, of 100, is the number to which 10 must be raised to get 100, i.e. 2.

- The number 2, raised to the power 3, is equal to 8, i.e. $2^3 = 8$.
  We say that the logarithm to the base 2, of 8, is 3. i.e. $\log_2 8 = 3$.

  The logarithm to the base 2, of 8, is the number to which 2 must be raised to get 8, i.e. 3.

- The number 5, raised to the power –1, is equal to 0.2, i.e. $5^{-1} = 0.2$.
  We say that the logarithm to the base 5, of 0.2, is –1. i.e. $\log_5 0.2 = -1$.

  The logarithm to the base 5, of 0.2, is the number to which 5 must be raised to get 0.2, i.e. –1.

To generalise: For some *positive* number a,

$$\text{If } a^x = b \text{ then } \log_a b = x.$$

i.e. The logarithm to the base a, of b, is the number to which a must be raised to give b, (x in this case).

For example:

$$\log_{10} 1000 = 3$$

(Because 1000 is ten to the power **3**.)

$$\log_{10} 10\,000 = 4$$

(Because 10 000 is ten to the power **4**.)

$$\log_2 32 = 5$$

(Because 32 is two to the power **5**.)

$$\log_2 0.5 = -1$$

(Because 0.5 is two to the power **–1**.)

$$\log_4 16 = 2$$

(Because 16 is four to the power **2**.)

$$\log_4 2 = 0.5$$

(Because 2 is four to the power **0.5**.)

$\log_{10}(1000)$
3
$\log_{10}(10000)$
4
$\log_2(32)$
5
$\log_2(0.5)$
–1
$\log_4(16)$
2
$\log_4(2)$
0.5

The statement $\log_a b = x$ is the logarithmic equivalent of the exponential statement $a^x = b$ (which could be written as $a^{\log_a b} = b$).

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Note: • In the previous examples the numbers involved were such that the logarithms could be determined mentally. If this is not the case, for example $\log_{10} 50$ or perhaps $\log_2 17$, some calculators can evaluate such expressions directly.

$\log_{10}(50)$	
	1.698970004
$\log_2(17)$	
	4.087462841

• To avoid having to write the base of the logarithm every time, we can omit the '10' for logarithms to base ten.

Thus $\log 50$, with no specific base indicated, is taken as $\log_{10} 50$.

Other bases need to be clearly indicated.

$\log_{10}(50)$	
	1.698970004
$\log(50)$	
	1.698970004

• On the previous page it was stated that:

For some *positive* number a,

$$\text{If } a^x = b \text{ then } \log_a b = x.$$

If we require a to be positive (i.e. we cannot have the logarithm with a negative base) it follows that b must also be positive (because if a is positive it follows that a to some power must also be positive).

I.e., we cannot determine the logarithm of a negative number.

(If asked to determine the logarithm of a negative number, say $\log(-3)$, some calculators will indicate 'error' whilst others may give a 'complex' number. Whilst students of *Mathematics Specialist* will be familiar with the idea of complex numbers, they are beyond the scope of this *Mathematics Methods* unit. As far as this unit is concerned we cannot determine the logarithm of a negative number.)

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EXAMPLE 1

Without the assistance of a calculator find

a $\log_2 16$ **b** $\log_4 8$ **c** $\log(0.1)$

then use a calculator to confirm your answers.

Solution

a
$$\begin{array}{llcl} \text{Let} & \log_2 16 & = & x \\ \text{then} & 2^x & = & 16 \\ \text{i.e.} & 2^x & = & 2^4 \\ \text{giving} & x & = & 4 \\ \therefore & \log_2 16 & = & 4 \end{array}$$

b
$$\begin{array}{llcl} \text{Let} & \log_4 8 & = & x \\ \text{then} & 4^x & = & 8 \\ \text{i.e.} & 2^{2x} & = & 2^3 \\ \text{giving} & x & = & 1.5 \\ \therefore & \log_4 8 & = & 1.5 \end{array}$$

c
$$\begin{array}{llcl} \text{Let} & \log(0.1) & = & x \\ \text{then} & 10^x & = & 0.1 \\ \text{i.e.} & 10^x & = & 10^{-1} \\ \text{giving} & x & = & -1 \\ \therefore & \log(0.1) & = & -1 \end{array}$$

$\log_2(16)$	
	4
$\log_4(8)$	
	1.5
$\log(0.1)$	
	–1

Note: (For interest only.) Prior to the ready access of calculators, logarithms were commonly used as an aid to calculation.

Evaluating a product like 216.4×171.2 or a quotient like $136.5 \div 16.5$ now takes only a few seconds but how would you cope with these calculations without a calculator?

In earlier centuries, mathematicians worked to produce tables of logarithms to the base ten. I.e. They produced conversion tables that allowed a number (say 2.884) to be expressed as a power of ten ($\sim 10^{0.46}$).

When two numbers had to be multiplied (or divided) these tables were used to convert each to powers of ten, these powers were then added (or subtracted) and other tables were then used to convert these powers of ten back to give the answer to the question. Though we no longer need to use logarithms in this way the $\boxed{\log}$ button on our calculator does hold the conversion information.

From your calculator $\log_{10} 500 \approx 2.69897$ i.e. $10^{2.69897} \approx 500$,

$\log_{10} 0.24 \approx -0.6198$ i.e. $10^{-0.6198} \approx 0.24$.

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Exercise 1A

Write each of the following as exponential statements.

1 $\log_2 8 = 3$

2 $\log_7 49 = 2$

3 $\log_{49} 7 = 0.5$

4 $\log_{10} 1000 = 3$

5 $\log_5 625 = 4$

6 $\log_4 32 = 2.5$

7 $\log_5 (0.04) = -2$

8 $\log_3 \left(\frac{1}{9}\right) = -2$

9 $\log_a x = y$

10 $\log_b y = c$

11 $\log_x a = p$

12 $\log_a x = 3$

13 $\log_3 5 = y$

14 $\log_2 3 = x$

15 $\log_x 5 = 4$

16 $\log_3 5 = p$

Write each of the following as logarithmic statements.

17 $2^6 = 64$

18 $3^4 = 81$

19 $81 = 9^2$

20 $9^{\frac{3}{2}} = 27$

21 $2^{-1} = 0.5$

22 $2^{-2} = 0.25$

23 $10^2 = 100$

24 $10^{-2} = 0.01$

25 $r = p^q$

26 $r^p = q$

27 $2^x = y$

28 $3^y = z$

29 $5^k = 4$

30 $7^y = 3$

31 $7 = 3^p$

32 $x = e^y$

Without the assistance of a calculator evaluate each of the following.

33 $\log_8 64$

34 $\log_2 128$

35 $\log 10\,000$

36 $\log_3 243$

37 $\log_2 \left(\frac{1}{2}\right)$

38 $\log_2 \left(\frac{1}{16}\right)$

39 $\log_6 \left(\frac{1}{216}\right)$

40 $\log_2 (0.125)$

41 $\log_9 243$

42 $\log (0.001)$

43 $\log_6 6$

44 $\log_7 1$

45 $\log_a 1$

46 $\log_4 32$

47 $\log_a a$

48 $\log_a (a^3)$

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Use a calculator to evaluate each of the following, correct to three decimal places if rounding is necessary.

49 $\log 5$

50 $\log 25$

51 $\log 7$

52 $\log 49$

53 $\log 20$

54 $\log(7+3)$

55 $\log 7 + \log 3$

56 $\log 50 - \log 5$

57 If $\log_{10} b = c$
a can c be negative?
b can b be negative?

Laws of logarithms

WS Logarithm laws

Suppose that $\log_a b = x$ and $\log_a c = y$

It then follows that $b = a^x$ and $c = a^y$

But, we know that $a^x a^y = a^{x+y}$ and $a^x \div a^y = a^{x-y}$

i.e. $bc = a^{x+y}$ and $b \div c = a^{x-y}$

Writing these as logarithmic statements:

$$\log_a(bc) = x + y \quad \text{and} \quad \log_a\left(\frac{b}{c}\right) = x - y$$

Thus $\boxed{\log_a(bc) = \log_a b + \log_a c}$ and $\boxed{\log_a\left(\frac{b}{c}\right) = \log_a b - \log_a c}$

Again suppose $\log_a b = x$ from which it follows that $b = a^x$.

From the index laws we know that $(a^x)^n = a^{xn}$

i.e. $b^n = a^{xn}$

Writing this as a logarithmic statement:

$$\log_a(b^n) = nx$$

$$\boxed{\log_a(b^n) = n\log_a b}$$

Note also that from $a^1 = a$ it follows that $\boxed{\log_a a = 1}$

from $a^0 = 1$ it follows that $\boxed{\log_a 1 = 0}$

and from $\log_a(b^n) = n\log_a b$ it follows that $\boxed{\log_a\left(\frac{1}{b}\right) = -\log_a b}$

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EXAMPLE 2

Express each of the following as single logarithms.

a $\log x + \log y - 3\log z$

b $\log x + 1 - \log y$

Solution

a
$$\begin{aligned}\log x + \log y - 3\log z &= \log(xy) - \log(z^3)\\ &= \log\left(\frac{xy}{z^3}\right)\end{aligned}$$

b
$$\begin{aligned}\log x + 1 - \log y &= \log x + \log 10 - \log y\\ &= \log\left(\frac{10x}{y}\right)\end{aligned}$$

EXAMPLE 3

Without the assistance of a calculator, simplify $\log_2 12 + \log_2 36 - 3\log_2 3$.

Solution

$$\begin{aligned}\log_2 12 + \log_2 36 - 3\log_2 3 &= \log_2 12 + \log_2 36 - \log_2 3^3\\ &= \log_2\left(\frac{12\times 36}{27}\right)\\ &= \log_2 16\\ &= \log_2(2^4)\\ &= 4\log_2 2\\ &= 4\end{aligned}$$

EXAMPLE 4

If $\log_a 4 = p$ and $\log_a 5 = q$, express each of the following in terms of p and q.

a $\log_a 20$

b $\log_a 0.8$

c $\log_a(100a^2)$

Solution

a
$$\begin{aligned}\log_a 20 &= \log_a(4\times 5)\\ &= \log_a 4 + \log_a 5\\ &= p + q\end{aligned}$$

b
$$\begin{aligned}\log_a 0.8 &= \log_a(4\div 5)\\ &= \log_a 4 - \log_a 5\\ &= p - q\end{aligned}$$

c
$$\begin{aligned}\log_a(100a^2) &= \log_a 100 + \log_a(a^2)\\ &= \log_a(4\times 25) + 2\log_a a\\ &= \log_a 4 + \log_a 5^2 + 2\\ &= \log_a 4 + 2\log_a 5 + 2\\ &= p + 2q + 2\end{aligned}$$

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Exercise 1B

Express each of the following as a single logarithm.

1 $\log x + \log z$

2 $2\log x + \log y$

3 $2\log x + 3\log y$

4 $2\log x - \log y$

5 $\log a + \log b - \log c$

6 $3\log a + 4\log b - 2\log c$

7 $5\log c - \log(c^3) + \log a$

8 $2 + \log x$

9 $3 - (\log x + \log y)$

10 $3 - \log x + \log y$

Evaluate each of the following (without the use of a calculator).

11 $\log_2 24 - \log_2 3$

12 $\log_2 20 + \log_2 8 - \log_2 10$

13 $\log(10^4) - \log 10$

14 $\log_a(a^3) + \log_b(b^2) - \log_c(c^4)$

15 $\log_3 45 + 2\log_3 2 - \log_3 20$

16 $\log_3 4 - 2\log_3 6 - 2$

17 $\log 5 - (\log 2 + 2\log 5)$

18 $\log_a b + 2\log_a(ab) - 3\log_a b$

19 $\dfrac{\log_a b^3}{2\log_a b}$

20 $\dfrac{\log_a 48 - \log_a 3}{\log_a 2}$

21 If $\log_a 2 = p$ and $\log_a 3 = q$, express each of the following in terms of p or q or both p and q.

a $\log_a 6$

b $\log_a 18$

c $\log_a 12$

d $\log_a 0.\overline{6}$

e $\log_a(9a^4)$

f $\log a\ 0.\overline{2}$

22 If $\log_5 7 = a$ and $\log_5 2 = b$, express each of the following in terms of a or b or both a and b.

a $\log_5 49$

b $\log_5 28$

c $\log_5 1.75$

d $\log_5 50$

e $\log_5 490$

f $\log_5 700$

Find y in terms of x (and a if necessary) for each of the following.

23 $\log_a y = x$

24 $\log_a y = \log_a(2x)$

25 $\log_a y = 3\log_a x$

26 $2\log_a y = 3\log_a x$

27 $\log_a y = \log_a a + \log_a x$

28 $\log_a y = 2 + \log_a x$

29 $\log_a y = -\log_a x$

30 $\log_a y + \log_a x = 2$

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31 An experiment was conducted to test the rate at which students forget work. The students were tested on a particular period in History they had recently studied. They were then given repeat tests of a similar nature after that. In each test the group average was calculated. The results obtained seemed to fit fairly well with the rule:

Average score, t fortnights after the initial test $= 75 - 35\log(t+1)$

- **a** What was the average score in the initial test?
- **b** What was the average score four weeks after the initial test?
- **c** What was the average score eight weeks after the initial test?
- **d** How many fortnights after the initial test did the average score fall to 40?

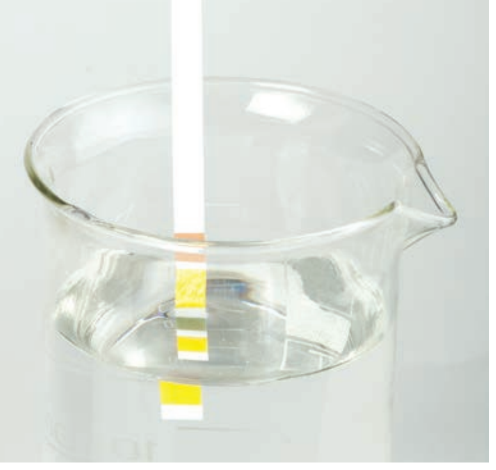
32 The Richter scale reading, R, of an earthquake of intensity I is given by

$$R = \log\left(\frac{I}{I_0}\right)$$

where I_0 is a minimum intensity level used for comparison.

- **a** Find R for an earthquake with intensity $1000\ I_0$.
- **b** An earthquake registers 5.4 on the Richter scale. Express its intensity in terms of I_0.
- **c** An earthquake measuring 6 on the Richter scale is how many times as intense as that of one measuring 5 on the Richter scale?
- **d** An earthquake measuring 7.7 on the Richter scale is how many times as intense as that of one measuring 5.9 on the Richter scale?

33 The acidity or alkalinity of a solution is measured by its pH. This is the negative of the logarithm to the base ten of the hydrogen ion concentration in moles per litre.



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The letters pH stand for *potential of hydrogen*.

Thus pH $= -\log_{10}$ (hydrogen ion concentration in moles per litre).

A pH below 7 indicates that a solution is acidic and a pH above 7 indicates alkaline. Values are usually between 0 and 14.

The pH of natural water is generally about 6 because dissolved carbon dioxide from the air forms carbonic acid.

Find the pH of each of the following:

- **a** Grapes. Hydrogen ion concentration 0.0001 moles/litre.
- **b** Beer. Hydrogen ion concentration 0.000 031 6 moles/litre.
- **c** Urine. Hydrogen ion concentration 0.000 000 25 moles/litre.
- **d** Eggs. Hydrogen ion concentration 0.000 000 016 moles/litre.
- **e** Blood. Hydrogen ion concentration 0.000 000 042 moles/litre.

If a solution has a pH of 5.25 what is its hydrogen ion concentration?

ISBN 9780170395205

34 Sound loudness is measured by comparing the intensity of the sound with the intensity of a sound that is just detectable by the human ear.

With L, the loudness of the sound in decibels (dB),
I, the intensity of the sound
and I_0, the intensity of sound just audible to the human ear, then

$$L = 10\log_{10}\left(\frac{I}{I_0}\right)$$

a If the noise level in a room was 40 dB it would be considered quiet.
Express the intensity of sound in this quiet room in terms of I_0.

b If the noise level in a room was 70 dB it would be considered noisy.
Express the intensity of sound in this noisy room in terms of I_0.

c How many times is the intensity of a 90 dB noise level that of the intensity of a 20 dB noise level?

Using logarithms to solve equations

Now that we know what logarithms are, and what rules they obey, can we use them to solve equations like

$$2^x = 23?$$

As mentioned earlier, we already have other methods for solving equations of this type but we include the logarithmic approach here because, as well as being useful functions in their own right, logarithms give us a method that can be quick to apply and that allows us to state an *exact* solution to the equation.

EXAMPLE 5

Use logarithms to solve the following equations, giving **exact** answers involving base ten logarithms.

a $2^x = 23$

b $2^{5x-1} = 3^x$

Solution

a

$$2^x = 23$$

Taking logs of both sides

$$\begin{aligned} \log(2^x) &= \log 23 \\ \therefore\quad x\log 2 &= \log 23 \\ x &= \frac{\log 23}{\log 2} \end{aligned}$$

b

$$2^{5x-1} = 3^x$$

Taking logs of both sides

$$\begin{aligned} \log(2^{5x-1}) &= \log(3^x) \\ (5x-1)\log 2 &= x\log 3 \\ 5x\log 2 - \log 2 &= x\log 3 \\ x(5\log 2 - \log 3) &= \log 2 \\ \therefore\quad x &= \frac{\log 2}{5\log 2 - \log 3} \end{aligned}$$

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Note:
- When 'taking logs of both sides' in the previous example we chose to use base ten logarithms because the question asked us to give our exact answer involving base ten logarithms. Without this requirement we could use any base of logarithm. Indeed a calculator, set the task of finding the exact solutions to the above equations, may well use *natural logarithms*, for which the abbreviation is 'ln'. We will consider the concept of natural logarithms later in this chapter.
- The answer for part **b** of the previous example could be written in a number of different ways, for example:

$$\frac{\log 2}{\log(2^5) - \log 3}, \qquad \frac{\log 2}{\log 32 - \log 3}, \qquad \frac{\log 2}{\log\left(\frac{32}{3}\right)}, \qquad \frac{-\log 2}{\log\left(\frac{3}{32}\right)}, \qquad \ldots$$

Hence do not be too quick to mark your answer wrong just because it appears different to the one given in the back of the book.

Exercise 1C

Solve each of the following, giving your answers in **exact** form involving logarithms to the base ten.

1 $3^x = 7$

2 $7^x = 1000$

3 $10^x = 27$

4 $2^x = 11$

5 $3^x = 17$

6 $7^x = 80$

7 $5^x = 21$

8 $10^x = 15$

9 $2^x = 70$

10 $6^{x+2} = 17$

11 $3^{x+1} = 51$

12 $8^{x-1} = 7$

13 $5^{x-1} = 3^{2x}$

14 $2^{x+1} = 3^x$

15 $4^{3x} = 5^{x+2}$

16 $3^{2x+1} = 2^{3x-1}$

17 $5(2^x) = 3 - 2^{x+2}$

18 $5^x + 4\,(5^{x+1}) = 63$

Use the substitution $y = 2^x$ to solve each of the following equations, giving your answers in exact form involving logarithms to the base ten.

19 $(2^x)^2 + 3(2^x) - 18 = 0$

20 $2^{2x} - 2^{x+3} + 15 = 0$

21 If $x = \log_2 7$ find an exact expression for x involving base ten logarithms.

Hint: Write the equation in exponential form and then take the logarithm of both sides.

22 Without the assistance of your calculator, find exact expressions involving base ten logarithms for each of the following. (Hint: Do question **21** first.)

a $\log_3 5$ **b** $\log_2 12$ **c** $\log_9 15$

d $\log_9 4$ **e** $\log_{2.5}(6.8)$ **f** $\log_{5.4}(9)$

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Applied questions

Solve the following questions **without** using the ability of your calculator to solve equations.

23 A strip of metal is to be made into a thin sheet by repeatedly passing it through a pair of compression rollers. The thickness, T, of the metal after it has passed through the rollers n times is given by

$$T = T_0\,(0.92)^n,$$

where T_0 is the initial thickness of the metal before any rolling has been done.

How many times should the metal be passed through the rollers if we require the final thickness to be as close as possible to, but *thinner* than, 20% of the initial thickness?

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24 A group of 200 insects were monitored in a laboratory experiment and the population was found to grow such that the number, N, present t days after the experiment commenced, approximately fitted the model

$$N = 200\,(2.7)^{0.1t}.$$

Find **a** the number of insects in the group 3 days after the experiment commenced,

b the number of insects in the group 5 days after the experiment commenced,

c on which day the population first exceeded 1000.

25 A driver with a high blood alcohol level is more likely to have an accident than is a driver with a low, or zero, blood alcohol level. If R% is the likelihood or risk of an accident, and a% is the blood alcohol level then let us suppose that the rule

$$R = (2.8)^{20a},$$

for $a \geq 0$, is a reasonable mathematical description of what seems to be the case.

For what value of a, the percentage blood alcohol level, is the risk of an accident 51%?

26 A company expects the weekly sales of a particular chocolate bar to increase from the usual 100 000 bars to 250 000 bars whilst their new advertising campaign is running. However, market research indicates that t weeks after the campaign finishes the weekly figures will have fallen to N bars, where

$$N = 100\,000 + 150\,000\,(1.1)^{-0.8t}.$$

If this predicted model proves to be correct, what will be the weekly sales figures

a 4 weeks after the campaign ceases?

b 8 weeks after the campaign ceases?

The company plans to repeat the campaign when sales fall to 135 000 bars per week. Approximately how many weeks after the first campaign ceases will this happen?

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27 If \$10 000 is invested at an interest rate of 8% per annum, compounded annually, the investment will grow to \$$P$ after x years where

$$P = 10\,000(1.08)^x.$$

a Find P after 3 years.

b Find P after 7 years.

c How long will it take (to the nearest year) for the investment to grow to \$50 000?

d How long will it take (to the nearest year) for the investment to grow to \$50 000 if

  i the interest throughout is 10% rather than 8%?

  ii the interest is 14% for the first 8 years and 10% thereafter?

e Find the annual interest rate necessary for the \$10 000 to double in value in 5 years. (Give your answer as a percentage, correct to one decimal place.)

Natural logarithms

Natural logarithms, a term mentioned a few pages earlier, are logarithms to the base 'e'. With e being a naturally occurring base in exponential equations describing growth and decay situations, it follows that it could well be a useful base to use with logarithms.

$$\text{If} \qquad b = e^x \qquad \text{then} \qquad x = \log_e b.$$

Note:
- We call logarithms to the base e *natural logarithms*.
- $\log_e x$ can be written as $\ln x$. (In this text we will use both forms.)
- Some calculators have two built-in logarithm options: $\boxed{\log}$ for $\log_{10} x$ and $\boxed{\ln}$ for $\log_e x$.

  Use your calculator to confirm that:

  $$\ln 1 = 0, \qquad \ln 2 \approx 0.693, \qquad \ln 10.5 \approx 2.351, \qquad \ln 2.718\,28 \approx 1.$$

- Whilst e and 10 are the more common bases for logarithms (indeed base ten logarithms are sometimes referred to as *common logarithms*), we have already seen that other bases are possible. In some cases logarithms to other bases may be readily evaluated, for example,

  $$\log_2 8 = 3 \qquad \text{and} \qquad \log_5 25 = 2.$$

  In other cases, for example,

  $$\log_2 7 \qquad \text{or} \qquad \log_5 0.6,$$

  we might use the ability of some calculators to evaluate the logarithm.

  $$\log_2 7 \approx 2.807, \qquad \log_5 0.6 \approx -0.317.$$

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- It is possible to express a logarithm in one base as an expression involving logarithms to another base. Indeed question 21 of the previous exercise asked you to do this when it asked you to express $\log_2 7$ in terms of base ten logarithms.

  Applying the technique suggested in that question to the general case gives us a *change of base formula*, as shown below.

If $x = \log_a b$ then	a^x	$=$	b
Taking logarithms	$\log_c (a^x)$	$=$	$\log_c b$
$\therefore$	$x \log_c a$	$=$	$\log_c b$
Giving	x	$=$	$\dfrac{\log_c b}{\log_c a}$

  Hence $$\boxed{\log_a b = \dfrac{\log_c b}{\log_c a}}$$ the **change of base formula**.

  Some calculators, if working in exact mode, will, when given the logarithm in a base other than 10 or e, display the answer in terms of natural logarithms.

  $\log_2(7)$

  $\dfrac{\ln(7)}{\ln(2)}$

  $\log_5(0.6)$

  $\dfrac{-(\ln(5) - \ln(3))}{\ln(5)}$

EXAMPLE 6

Use logarithms to solve the equation $e^{x+1} = 5$ giving your answer

a exactly,

and **b** correct to four decimal places.

Solution

a With the equation involving e it makes sense to use natural logarithms rather than logarithms to base ten.

$$
\begin{aligned}
& & e^{x+1} &= 5 \\
&\therefore & (x+1)\log_e e &= \log_e 5 \\
&\text{Thus} & x + 1 &= \log_e 5 \\
& & x &= \ln 5 - 1 \qquad \text{is the exact solution.}
\end{aligned}
$$

b $\therefore \quad x = 0.6094$ is the solution correct to 4 decimal places.

The reader should confirm that 0.6094 is also obtained if base ten logarithms are used instead of natural logarithms.

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Exercise 1D

Evaluate each of the following without the use of a calculator.

1 $\log_e e$

2 $\log_e\left(\frac{1}{e}\right)$

3 $\log_e(e^3)$

4 $\log_e \sqrt{e}$

5 $\ln \sqrt[3]{e}$

6 $\ln\left(\frac{1}{\sqrt{e}}\right)$

7 $\ln(e^{-3})$

8 $\ln\left(\frac{1}{\sqrt[3]{e}}\right)$

Clearly showing your use of natural logarithms, solve each of the following equations giving your answers as exact values.

9 $e^{x+1} = 7$

10 $e^{x+3} = 50$

11 $e^{x-3} = 100$

12 $e^{2x+1} = 15$

13 $5e^{3x-1} = 3000$

14 $4e^{x+2} + 3e^{x+2} = 7000$

15 $e^{2x} - 30e^x = -200$ (Hint: let $y = e^x$.)

Express each of the following in terms of natural logarithms and prime numbers (without the assistance of a calculator).

16 $\log_7 2$

17 $\log_2 21$

18 $\log_3 200$

19 $\log_5 50$

20 $\log_6 9$

21 $\log_9 6$

22 $\log_4 300$

23 $\log_8 220$

24 If $A = 2000e^{-t}$ find an exact expression for t in terms of A and evaluate it, correct to three decimal places, for
a $A = 1500$,
b $A = 500$,
c $A = 50$.

25 The population of a particular country was thought to be 22 300 000 in 2010.

Figures suggest that the population is growing such that, t years after 2010, it will be approximately P, where $P = 22\,300\,000e^{0.02t}$.

If the population growth continues as suggested, in which year would the population of this country reach
a 32 000 000?
b 45 000 000?

26 A certain culture of bacteria grows in such a way that t days after observation commences the number of bacteria present, N, is given by:

$$N \approx 5000e^{0.55t}.$$

According to this rule how many days after observation commences, to the nearest day, would the number of bacteria be
a 80 thousand?
b 750 thousand?

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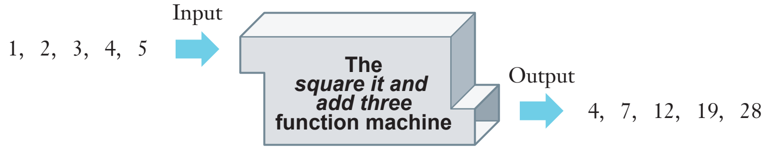
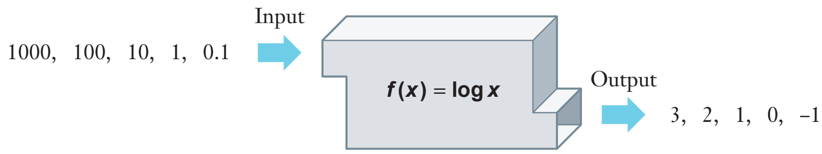
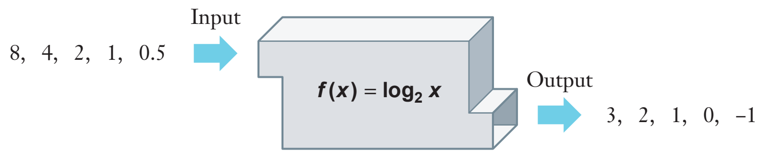
Logarithmic functions

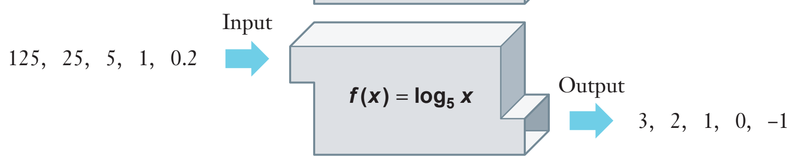
The *Preliminary work* mentioned that it can be helpful to view a function as a machine with a specific output for each given input:



Applying this idea to the concept of logarithms:







Graphs of logarithmic functions

What do the graphs of $y = \log x$, $y = \ln x$, $y = \log_5 x$, $y = \log_2 x$, etc., look like?

As the *Preliminary work* mentioned, it is anticipated that you are familiar with how the graph of $y = af[b(x - c)] + d$, differs from that of $y = f(x)$.

Use your calculator to view the graphs of $y = \log_a x$ for various values of $a > 0$.

WS Plotting log functions

In particular, starting with $y = f(x)$:

- Multiplying the right hand side of the equation by 'a' stretches (dilates) the graph parallel to the y-axis with scale factor 'a'. If 'a' is negative the graph is also reflected in the x-axis.
- Replacing x by bx dilates the graph parallel to the x-axis with a scale factor of $\frac{1}{b}$.
- Replacing x by $x - c$ translates the graph c units to the right. (If c is negative the translation is to the left).
- Adding 'd' to the right hand side of the equation translates the graph d units vertically upwards. (If d is negative the translation is vertically downwards.)

INVESTIGATE

Are these same effects evident when we consider the graphs of logarithmic functions?

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Exercise 1E

1 Determine the coordinates of the point where the graph of $y = \log_2(x + 8)$ cuts

a the x-axis, **b** the y-axis.

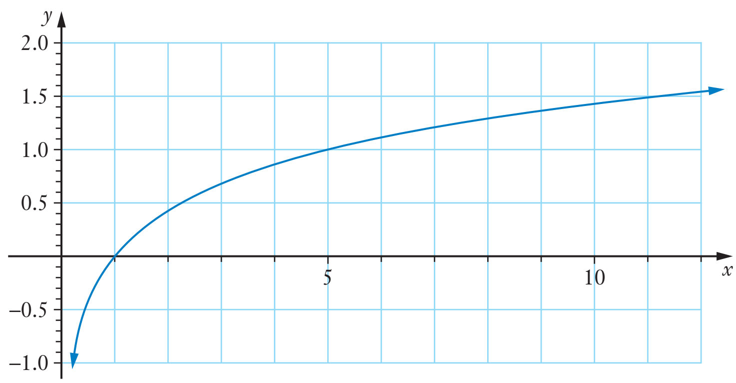
2 What are the coordinates of the point that is common to *all* graphs of the form $y = \log_p x$?

3 Find the coordinates of the point where the graph of $y = \log_a x$ cuts the line $y = 1$.

4 What is the equation of the vertical asymptote of the graph of

a $y = \log_p x$? **b** $y = \log_p(x - 3)$? **c** $y = \log_p x - 3$?

5 The graph below shows $y = \log_5 x$.



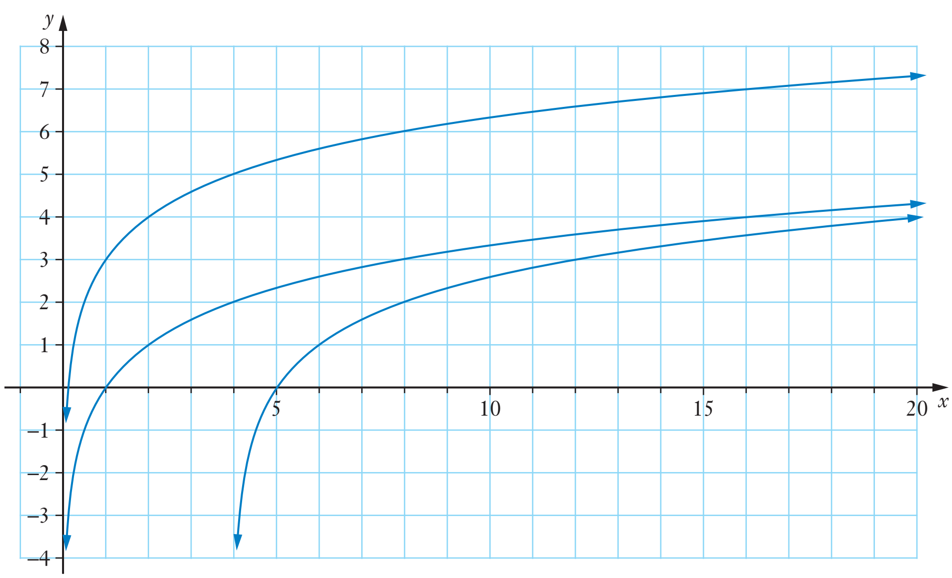
Use the graph to determine approximate solutions to each of the following.

a $\log_5 x = 0.5$, **b** $\log_5 x = 1.5$, **c** $x - 5^{0.8} = 0$, **d** $\log_5(x - 1) = 1.3$.

e Now solve each of the equations algebraically, with the assistance of your calculator to evaluate powers, giving answers rounded to 3 decimal places.

6 The graph below shows $y = \log_a x$, $y = \log_a(x - b)$, and $y = \log_a x + c$.

Determine the values of a, b and c given that they are all positive integers.



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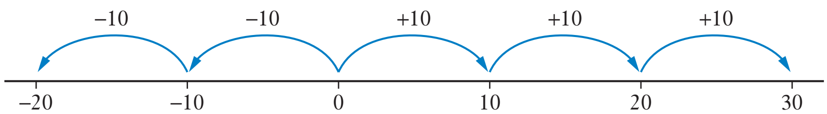
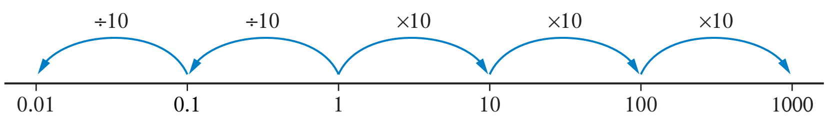
Logarithmic scale

The number line below shows a linear scale.

Suppose we start at a particular number location on this line. If moving a particular distance to the right (or left), increases (or decreases) the number we are located at by, say, 10, then on this linear scale all such movements of this size will increase (or decrease) the number we are located at by 10.



However, on a logarithmic scale, if moving a particular distance to the right (or left) multiplies (or divides) the number we are located at by, say, 10, then all such movements to the right (or left) will multiply (or divide) by 10.



In this way, in a logarithmic scale, the distance between consecutive powers of ten is constant.

Notice that the logarithmic scale shown above displays the numbers 1 to 1000 in the space that the linear scale at the top of the page displayed just zero to 30. This ability to display a greater range in the same space is one feature that makes logarithmic scales useful. Consider again Situation One encountered at the beginning of this chapter, for example. It would be difficult on a linear scale to show both the comparatively small world population of one million and the much larger current population of more than seven billion. Use of a logarithmic scale may solve this problem.

Before the ready availability of electronic calculators a device called a slide rule was a helpful aid when performing calculations. The slide rule was marked using a logarithmic scale rather than a linear scale. By placing two such scales with the same base together, and adding length a to length b, the fact that

$$\log_k a + \log_k b = \log_k (ab)$$

means that the combined length gives the product $a \times b$.

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Graphs with logarithmic scales

Some graph paper have a logarithmic scale on one axis (log-linear graph paper) or on both axes (log-log paper).

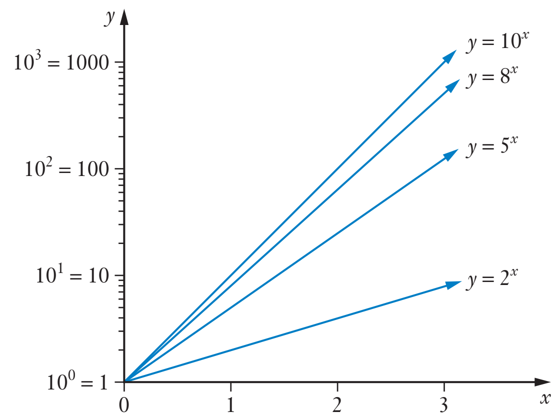
The log-linear graph on the right has a logarithmic scale on the y-axis.

On this graph, functions with equations of the form

$$y = a^x$$

will appear as straight lines

(as indeed will all functions of the form $y = ka^{bx}$, for k, a and b constants).



Use of logarithmic scales

As mentioned on the previous page, if we wish to display data that has a large range, a logarithmic scale can be useful.

We have also already seen in this chapter that this 'multiplication effect' of a logarithmic scale is used in the Richter scale of earthquake intensity, in the modelling of memory activity, in measuring the acidity or alkalinity of solutions (the pH value) and in sound measurement (decibels). Three of these applications are mentioned again below and also mention is made of the use of a logarithmic scale in musical scales.

The Richter scale

A seismograph is an instrument that measures vibrations from an earthquake graphically. The base ten logarithm of the amplitude of these measurements (corrected for the distance the seismograph is from the earthquake epicentre) gives the strength of the earthquake on the Richter scale.

The use of base ten logarithms means that for each unit increase on the Richter scale, the amplitude of the vibrations is multiplied by ten.

pH scale

The pH scale (potential of Hydrogen) is a measure of the acidity or alkalinity of a solution. This is the negative of the logarithm to the base ten of the hydrogen ion concentration in moles per litre.

A pH of 7 is regarded as neutral. Pure water is neutral, it is neither acidic nor alkaline. The pH of pure water is a reference point for acidity and alkalinity. A pH above 7 indicates a solution is alkaline, below 7 indicates the solution is acidic.

A solution with a pH of 3 is ten times as acidic as a solution with a pH of 4.

A solution with a pH of 10 is one hundred times as alkaline as a solution with a pH of 8.

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Scale of loudness

The decibel (dB) scale measures loudness and is based on multiples of ten. Hence this too is a logarithmic scale using base ten logarithms.


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Music scale

If one musical note has frequency f and another has frequency $2f$ the frequency ratio is said to be one octave. Thus, each time the frequency doubles we go up one octave. This use of powers again means that a logarithmic scale is used.

To determine how many doublings are involved in a change from a frequency f_1 to f_2 we solve

$$\frac{f_2}{f_1} = 2^x$$

Hence $\quad \log\left(\frac{f_2}{f_1}\right) = x\log 2 \quad$ giving $\quad x \approx 3.32\log\left(\frac{f_2}{f_1}\right).$

Exercise 1F

1 A particular scale measures N as a function of L according to the rule

$$N = -\log_{10}(2L)\ .$$

Find **a** N when $L = 3.2 \times 10^{-8}$,

b L when $N = 9.5$.

2 If x octaves are involved between a note of frequency f_1 hertz (Hz) and one of f_2 Hz then

$$x = \frac{1}{\log 2} \times \log\left(\frac{f_2}{f_1}\right).$$

a How many octaves are there between a frequency of 20 Hz to one of 50 Hz?

b If something has a frequency range of 3 octaves, and the lower frequency is f_1, express the higher frequency in terms of f_1.

3 The pH of a solution is defined as

$$\text{pH} = -\log(\text{H}^+),$$

where H^+ is the hydrogen ion concentration in moles per litre.

a Find H^+ for pure water, pH = 7.

b Find the pH for lemon juice, $\text{H}^+ = 0.01$ moles per litre.

4 The 'logit' function (pronounced *lowjit*) is used in some branches of probability and statistics. If p is the probability of an event occurring then

$$\text{logit}(p) = \ln\left(\frac{p}{1-p}\right).$$

a If $p = 0.2$ find $\text{logit}(p)$ giving your answer correct to two decimal places.

b If $\text{logit}(p) = 4$ find p giving your answer correct to two decimal places.

c If an event has a probability of occurring of p what is the significance of $\text{logit}(p)$ being negative?

d If, for real x and real k, $\ln\left(\frac{x}{1-x}\right) = k$, show that x will always be between zero and one, whatever the value of k.

5 Comment on the following statement:

The cost of the damage caused by an earthquake of Richter scale 7 is ten times that of one with Richter scale 6.

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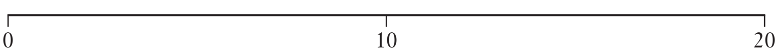
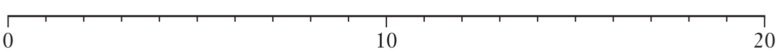
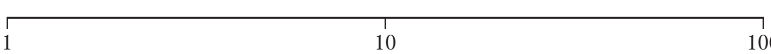
6 With the linear scale shown below indicating 0, 10 and 20,



we know where to mark 1, 2, 3, etc.:



However, with the logarithmic scale below, indicating 1, 10 and 100,



where would we mark 2, 4, 5, 20, 30, 50?

Try to draw such a logarithmic scale yourself with these numbers appropriately placed. (Use 5 cm for the distance from 1 to 10, and the distance from 10 to 100, i.e. use 5 cm to represent 'multiplication by 10'.)

(As mentioned on page 21, before the ready availability of electronic calculators a device called a slide rule used such a scale and was helpful in performing calculations.)

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Miscellaneous exercise one

Logarithms review

This miscellaneous exercise may include questions involving the work of this chapter and the ideas mentioned in the Preliminary work section at the beginning of the book.

Differentiate the following with respect to x.

1 $5x^3$

2 $x(x^2 + 1)$

3 $\dfrac{x - 3}{2x + 5}$

4 $(x^3 + 1)^4$

5 e^x

6 $2e^x$

7 $10e^x$

8 $e^x + 3x^2 + x^3$

9 e^{5x}

10 $3e^{4x}$

11 $3e^{2x}$

12 $2e^{3x} + 3e^{2x}$

Write each of the following as exponential statements.

13 $\log_3 81 = 4$

14 $\log_6 216 = 3$

15 $\log_2 (0.25) = -2$

16 $\log_a b = c$

17 $\log_a c = b$

18 $\log_b a = c$

19 $\log_c a = b$

20 $\log_x 2 = 5$

Write each of the following as logarithmic statements.

21 $2^3 = 8$

22 $25 = 5^2$

23 $4^{-1} = 0.25$

24 $2^{-3} = 0.125$

25 $7^x = y$

26 $a^2 = p$

27 $10^y = z$

28 $x = e^y$

Evaluate each of the following (without the assistance of a calculator).

29 $\log_2 32$

30 $\log_5 125$

31 $\log_{10} 10$

32 $\log 1000$

33 $5 + \ln e$

34 $4 - \ln e^2$

35 $6\ln\sqrt{e}$

36 $\log_2 8 + \ln\left(\frac{1}{e}\right)$

37 $\log_a 1$

38 $\log_a a$

39 $\log_a(a^3)$

40 $\log_a \sqrt{a}$

Use natural logarithms to solve each of the following equations, giving exact answers.

41 $e^{x+1} = 12$

42 $e^{x+2} = 25$

43 $e^{x-1} = 150$

44 $e^{2x+1} = 34$

45 $5e^{x+1} + 3e^{x+1} = 200$

46 $e^{2x} - 12e^{x} = -35$

Express each of the following as a single logarithm.

47 $3\log x + \log y$

48 $2\log x - 3\log y$

49 $2\log a + \log b - 3\log c$

50 $3 + \log x$

51 $2 + \ln x$

52 $3 - \ln x + 2\ln y$

53 A particular company required P tonnes of fossil fuel in 2010. Figures suggest that this annual requirement is increasing in such a way that t years after 2010 the company will require $Pe^{0.1t}$ tonnes. If this suggested rule is correct, by what year will the requirement for fossil fuel for this company be approximately five times its 2010 requirement?

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54 A body is initially at rest at an origin, O. It then moves in a straight line such that its acceleration, t seconds later, is $0.1e^{0.1t}$ m/s^2.

a Find the velocity of the body when $t = 10$.

b Find the displacement of the body from O when $t = 10$.

c Find a formula involving T for the distance the body travels from $t = T$ to $t = T + 1$.

Use your formula from **c** to determine, correct to 3 decimal places, the distance the body moves in

d the third second,

e the tenth second.

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